B.Sc Physics, Part-III

Paper-V, Group-A

Second solution of Bessel's equation:—

In the last lecture note on Bessel's equation, we have obtained the solution of Bessel's equation of order p (See last lecture note for details)

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p} \qquad (1)$$

where we have considered the case when $s = p$.

In the present class note, we will discuss the solution for the case when $s = -p$. Since we have already discussed the details of obtaining solution when $s = p$. Here we can simply replace $p \to -p$ in eqn-(1) to obtain the solution. Therefore

$$\boxed{J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p}} \qquad (2)$$

In the above expression, if p is not an integen, then J_p and J_{-p} are two independent solutions of Bessel's equation and linear combination of them is a general solution.

If p is integer $\rightarrow$ first few terms of J_{-p} are zero since $\Gamma(n-p+1)$ in denomenator is infinite ~~of negative argument of~~ for Γ of a negative integer. And $J_{-p}(x)$ starts with the term x^p instead of x^{-p}. Thus

$$\boxed{J_{-p}(x) = (-1)^p J_p(x)} \quad \text{--- (3)} \quad \text{for } p = \text{integer}$$

Thus for integral p, $J_{-p}(x)$ is not an independent solution.

H.W.

(1) Using equation (1) and (2) write down the first few terms of $J_0(x)$, $J_1(x)$, $J_{-1}(x)$, $J_{-2}(x)$, $J_2(x)$. Show that $J_{-1}(x) = -J_1(x)$ and $J_{-2}(x) = J_2(x)$.

(2) Show that $\sqrt{\frac{\pi x}{2}}\, J_{-\frac{1}{2}}(x) = \cos x$

(3) Show that $J_{\frac{3}{2}}(x) = x^{-1} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$.